SEPARATE ADJUSTMENT OF CLOSE RANGE PHOTOGRAMMETRIC MEASUREMENTS

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ABSTRACT

Photogrammetric methods will increasingly be used for real-time applications. For example, in a manufacturing environment the position of components must be located quickly and accurately for many assembly tasks. The computational effort should be minimal and if possible completely predictable. The conventional bundle adjustment method is unlikely to be used in this context because of the speed requirement while the direct linear transform method has modelling deficiencies for high precision measurement, furthermore, direct intersection with prior camera calibration does not allow for the dynamic situation found in industrial environments. The paper will describe a methodology for solving collinearity equations. Unlike the traditional bundle adjustment which solves for the unknown spatial co-ordinates of targets and camera parameters *simultaneously*, a *separate* solution for least squares estimation is developed which divides the parameters into two different groups, one for camera parameters, and the other for the co-ordinates of object points. With camera parameters fixed the 3-D co-ordinates of any spatial target can be located by spatial intersection of lines, and with spatial targets fixed the camera parameters can be determined by spatial resection. This process is repeated for both sets of parameters which are gradually refined. The final result can be proven to be statistically the same as would be achieved using the bundle adjustment but with a considerable time and memory saving. This separate adjustment method is found very successful to deal with close range photogrammetric measurements (CRPM), especial for a multistation convergent network.

1. SIMULTANEOUS LEAST SQUARES ADJUSTMENT

In surveying and close range photogrammetry, redundant measurements are always necessary for high precision, reliability and statistics (Mikhail & Gracie 1981, Cooper 1987). This means that the number of observation is more than the minimum for a unique solution of the unknown parameters. This section will briefly discuss the simultaneous least squares estimation for redundant measurements. Let the functional model be expressed as

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{l} \qquad (1.1)$$

where x is a vector of the unknown parameters and l is a vector of the observations. The linearized observation equations may be expressed as

$$\boldsymbol{A}\Delta\boldsymbol{x} = \boldsymbol{b} + \boldsymbol{v} \ : \boldsymbol{W} \qquad (1.2)$$

where $\boldsymbol{A} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}$ is a Jacobi matrix, $\boldsymbol{v}$ is a vector of residuals for the observations and $\boldsymbol{W}$ is the weight matrix of the observations. There are obviously many possible values for $\boldsymbol{v}_i$ to fit the functional model. Several methods exist to give a minimum value for the combinations of residuals (Kuang 1996). The least squares criterion is the most popular which minimises the sum of the weighted squares of the residuals. When all the unknown parameters are considered simultaneously the least squares estimation gives the following solution

$$\Delta\boldsymbol{x} = (\boldsymbol{A}^t\boldsymbol{W}\boldsymbol{A})^{-1}\boldsymbol{A}^t\boldsymbol{W}\boldsymbol{b} \qquad (1.3)$$

The cofactor matrix of the estimated parameters is given by

$$\boldsymbol{Q}_x = (\boldsymbol{A}^t\boldsymbol{W}\boldsymbol{A})^{-1} \qquad (1.4)$$

2. BUNDLE ADJUSTMENT OF CRPM

In close range photogrammetry, when m cameras are used to measure n object points, $\boldsymbol{x}$ will be a vector with $(3n+6m)$ unknown parameters and l will be a vector with $2mn$ image observations. The unknown parameters $\boldsymbol{x}$ can be divided into two groups, $\boldsymbol{x}_1$ for 3D coordinates of the object points and $\boldsymbol{x}_2$ for the camera parameters.

Therefore in equation (1.2) $\boldsymbol{A}$ and $\boldsymbol{Dx}$ become

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \end{bmatrix} \text{ and } \Delta\boldsymbol{x} = \begin{bmatrix} \Delta\boldsymbol{x}_1 \\ \Delta\boldsymbol{x}_2 \end{bmatrix}$$

where $\boldsymbol{A}_1 = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}_1}$ and $\boldsymbol{A}_2 = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}_2}$

The unknown parameters $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ may be solved simultaneously as follows

$$\Delta\boldsymbol{x} = \begin{bmatrix} \Delta\boldsymbol{x}_1 \\ \Delta\boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{A}_1^t\boldsymbol{W} \\ \boldsymbol{A}_2^t\boldsymbol{W} \end{bmatrix} \boldsymbol{b}$$

$$= \boldsymbol{N}^{-1}\begin{bmatrix}\boldsymbol{A}_1^t\boldsymbol{W}\\ \boldsymbol{A}_2^t\boldsymbol{W}\end{bmatrix}\boldsymbol{b} \qquad (2.1)$$

in which $\boldsymbol{A}_{ij} = \boldsymbol{A}_i^t\boldsymbol{W}\boldsymbol{A}_j$. The matrix to be inverted here has a dimension of (3n+6m)×(3n+6m). If four cameras are used to measure 100 object points, 3n+6m=324. The inverse of such a large matrix is computationally expensive and may not be achieved in real time without special measures.

One of the methods that might be used is inversion by partitioning (Frank Ayres 1962, Brown 1976, Granshaw 1980), which gives

$$\begin{aligned}\boldsymbol{N}^{-1} &= \begin{bmatrix}\boldsymbol{A}_{11} & \boldsymbol{A}_{12}\\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22}\end{bmatrix}^{-1}\\ &= \begin{bmatrix}\boldsymbol{B}_{11} & \boldsymbol{B}_{12}\\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22}\end{bmatrix}\end{aligned} \qquad (2.2)$$

in which

$$\begin{cases}\boldsymbol{B}_{11} = \boldsymbol{A}_{11}^{-1} + \boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12}\boldsymbol{K}^{-1}\boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1}\\ \boldsymbol{B}_{12} = -\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12}\boldsymbol{K}^{-1}\\ \boldsymbol{B}_{12} = \boldsymbol{B}_{21}^t\\ \boldsymbol{B}_{22} = \boldsymbol{K}^{-1}\end{cases} \qquad (2.3)$$

and

$$\boldsymbol{K} = \boldsymbol{A}_{22} - \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12} \qquad (2.4)$$

Since $\boldsymbol{A}_{11}$ is a block diagonal matrix, its inverse can be computed by inverting n 3×3 small matrices, which provides a big saving of time and memory. Matrix $\boldsymbol{K}$ is generally full with a size of 6m×6m. Therefore the inverse of $\boldsymbol{K}$ becomes the main cost of processing the matrices. An alternative to calculating the inverse of $\boldsymbol{N}$ is to calculate the $\boldsymbol{B}$ matrices by

$$\begin{cases}\boldsymbol{B}_{11} = \boldsymbol{K}^{-1}\\ \boldsymbol{B}_{12} = -\boldsymbol{K}^{-1}\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\\ \boldsymbol{B}_{21} = \boldsymbol{B}_{12}^t\\ \boldsymbol{B}_{22} = \boldsymbol{A}_{22}^{-1} + \boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21}\boldsymbol{K}^{-1}\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\end{cases} \qquad (2.5)$$

and

$$\boldsymbol{K} = \boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21} \qquad (2.6)$$

This time the inverse of $\boldsymbol{A}_{22}$ can easily be obtained by computing the inverse of a series of 6×6 small matrices. However the size of $\boldsymbol{K}$ is 3n×3n and it is generally full, hence the inverse of $\boldsymbol{K}$ could be expensive to calculate since 3n is normally much larger than 6m in close range photogrammetry. But it could be possible that 6m is larger than 3n in some special cases such as when hundreds of images are taken to measure only a few object points. In this case Eq (2.5) and (2.6) are more suitable for calculating the inverse of $\boldsymbol{N}$.

In real 3D measurement applications, the coordinates of the object points $\boldsymbol{x}_1$ are more important than the camera parameters $\boldsymbol{x}_2$. The corrections of $\boldsymbol{x}_1$ can be obtained by

$$\begin{aligned}\Delta\boldsymbol{x}_1 &= \begin{bmatrix}\boldsymbol{B}_{11} & \boldsymbol{B}_{12}\end{bmatrix}\begin{bmatrix}\boldsymbol{A}_1^t\\ \boldsymbol{A}_2^t\end{bmatrix}\boldsymbol{W}_l\boldsymbol{b}\\ &= (\boldsymbol{B}_{11}\boldsymbol{A}_1^t + \boldsymbol{B}_{12}\boldsymbol{A}_2^t)\boldsymbol{W}_l\boldsymbol{b}\end{aligned} \qquad (2.7)$$

and the cofactor matrix of the estimated coordinates is given by

$$\begin{aligned}\boldsymbol{Q}_{x_1} &= (\boldsymbol{B}_{11}\boldsymbol{A}_1^t + \boldsymbol{B}_{12}\boldsymbol{A}_2^t)\boldsymbol{W}_l\boldsymbol{Q}_b((\boldsymbol{B}_{11}\boldsymbol{A}_1^t + \boldsymbol{B}_{12}\boldsymbol{A}_2^t)\boldsymbol{W}_l)^t\\ &= (\boldsymbol{B}_{11}\boldsymbol{A}_1^t + \boldsymbol{B}_{12}\boldsymbol{A}_2^t)\boldsymbol{W}_l(\boldsymbol{A}_1\boldsymbol{B}_{11} + \boldsymbol{A}_2\boldsymbol{B}_{21})\\ &= \boldsymbol{B}_{11}\boldsymbol{A}_{11}\boldsymbol{B}_{11} + \boldsymbol{B}_{11}\boldsymbol{A}_{12}\boldsymbol{B}_{21} + \boldsymbol{B}_{12}\boldsymbol{A}_{21}\boldsymbol{B}_{11} + \boldsymbol{B}_{12}\boldsymbol{A}_{22}\boldsymbol{B}_{21}\end{aligned} \qquad (2.8)$$

The computational complexity of the last three terms is directly proportional to 6m×(3n)^2 provided that the right order is applied when products of matrices are calculated. The computational complexity of the first term is directly proportional to (3n)^3 if the products of the matrices are calculated directly. This is normally too expensive. The complexity can be reduced to 6m×(3n)^2 when $\boldsymbol{B}_{11}$ is replaced by Eq (2.3) and the right order is considered.

The foregoing discussion of the bundle adjustment is based on the assumption that the coefficient matrix $\boldsymbol{N}$ is non-singular. However, photogrammetric observations are obtained from images which do not include any information in the object space to define a datum. If control points are involved in the bundle adjustment the datum problem may be solved. Otherwise constraints must be applied to remove the rank defects of $\boldsymbol{N}$ to make it possible to estimate the unknown parameters. Inner constraints are often used for the unbiased free network adjustment, which can be applied to the object points or to the camera exterior parameters, or to both.

Inner constraints can be applied in the bundle adjustment to remove the rank defects of the design matrix, so the unknown parameters can be estimated. The datum is defined by the starting values. Because the starting values are arbitrary, the estimated results are in an arbitrary datum. Different starting values lead to different results. But they are equivalent in the sense of least squares and the shape of the object remains unchanged.

With the additional seven constraint equations, the total number of equations used for the (3n+6m) parameters is (2mn+7). The following conditions must be satisfied to enable the bundle adjustment to work, i.e.,

$$2\boldsymbol{mn} + 7 \geq 3\boldsymbol{n} + 6\boldsymbol{m} \qquad (2.9)$$

which gives

$$\boldsymbol{n} \geq 3 + \frac{2}{2\boldsymbol{m} - 3} \qquad (2.10a)$$

and

$$\boldsymbol{m} \geq 1.5 + \frac{1}{\boldsymbol{n} - 3} \qquad (2.10b)$$

This means that a minimum of four object points are required generally (when m≥3) and a minimum of five object points are needed if only two photographs are used. A minimum of two cameras are required generally (when n≥5) and a minimum of three cameras are needed if only four object points are involved. But actually three object points are enough to

determine the camera parameters (exterior parameters) provided that they all appear on the camera image plane and two cameras are adequate to solve the 3D coordinates of the object points. That implies enough information has been given to determine the unknown parameters with three object points and two cameras. But this does not apply to the situation of the inner constrained bundle adjustment. When inner constraints are applied, the special structure of $\boldsymbol{A}_{11}$ or $\boldsymbol{A}_{22}$ is spoiled. This makes the computation of the inverse of the coefficient matrix $\boldsymbol{N}$ more complicated.

3. SEPARATE LSE

The bundle adjustment is based on the simultaneous least squares methods. These methods have been widely used in surveying and photogrammetry for various purposes. However as far as speed is concern these methods are not ideal for real-time applications in close range photogrammetry. In this section an alternative method of least square estimation - separate least squares estimation - is introduced.

For a linear system, the functional model may be expressed as

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{3.1}$$

in which
$\boldsymbol{x} = (x_1, x_2, ..., x_u)^t$ is a vector of the unknown parameters,
$\boldsymbol{A}$ is a $m \times u$ coefficient matrix ($m>u$), and
$\boldsymbol{b}$ is a $m \times 1$ vector of the measured elements (observations).

To estimate the unknown parameters separately, $\boldsymbol{x}$ may be divided into k groups, i.e., $\boldsymbol{x} = (\boldsymbol{X}_1, \boldsymbol{X}_2, ..., \boldsymbol{X}_k)$ and $\boldsymbol{A}$ into $\boldsymbol{A} = (\boldsymbol{A}_1, \boldsymbol{A}_2, ..., \boldsymbol{A}_k)$ accordingly. Therefore Eq (3.1) becomes

$$\boldsymbol{A}_1\boldsymbol{X}_1 + \boldsymbol{A}_2\boldsymbol{X}_2 + \Lambda + \boldsymbol{A}_k\boldsymbol{X}_k = \boldsymbol{b} \tag{3.2}$$

Suppose there are q parameters in $\boldsymbol{X}_i$, so $\boldsymbol{A}_i$ will be a $m \times q$ matrix. When estimating $\boldsymbol{X}_i$, other parameters are considered as constants and the corresponding terms are moved to the right hand side of Eq (3.2), i.e.,

$$\boldsymbol{A}_i\boldsymbol{X}_i = \boldsymbol{b}_i \tag{3.3}$$

in which

$$\boldsymbol{b}_i = \boldsymbol{b} - \boldsymbol{A}_J\boldsymbol{X}_J \tag{3.4}$$

and

$$\boldsymbol{A}_J\boldsymbol{X}_J = \sum_{\substack{j=1 \\ j\neq i}}^{k} \boldsymbol{A}_j\boldsymbol{X}_j \tag{3.5}$$

Since $\boldsymbol{A}_i\boldsymbol{X}_j$ is non-stochastic, so $\boldsymbol{W}_{b_i} = \boldsymbol{W}_b$. By the linear least square estimation $\boldsymbol{X}_i$ is solved as

$$\begin{aligned}\boldsymbol{X}_i &= (\boldsymbol{A}_i^t\boldsymbol{W}_b\boldsymbol{A}_i)^{-1}\boldsymbol{A}_i^t\boldsymbol{W}_b\boldsymbol{b}_i \\ &= \boldsymbol{C}_i + \boldsymbol{D}_i\boldsymbol{X}_J\end{aligned} \tag{3.6}$$

in which

$$\boldsymbol{C}_i = (\boldsymbol{A}_i^t\boldsymbol{W}_b\boldsymbol{A}_i)^{-1}\boldsymbol{A}_i^t\boldsymbol{W}_b\boldsymbol{b} \tag{3.7}$$

and

$$\boldsymbol{D}_i = -(\boldsymbol{A}_i^t\boldsymbol{W}_b\boldsymbol{A}_i)^{-1}\boldsymbol{A}_i^t\boldsymbol{W}_b\boldsymbol{A}_J \tag{3.8}$$

The size of $\boldsymbol{A}_i^t\boldsymbol{W}_B\boldsymbol{A}_i$ is $q \times q$, much smaller than that of $\boldsymbol{A}^t\boldsymbol{W}_b\boldsymbol{A}$, which is $m \times m$ for the simultaneous LSE. This is why time and memory are saved. After $\boldsymbol{X}_i$ is solved, it is considered constant and used to solve the other parameters. The constants assigned to the unknown parameters may be starting values or the least squares estimates from previous iterations. Based on these constants the least squares estimation of the current iteration may not be the final results unless several iterations have been applied. The iterative process stops when the corrections for all the parameters are less than a given significant value. It is noticed that $\boldsymbol{C}_i$ and $\boldsymbol{D}_i$ will remain unchanged during the iterative process in the linear case. Once computed for the first iteration, they can be used repeatedly to calculate the unknown parameters.

By the general law of propagation of covariance, the cofactor matrix of the estimated parameters $\boldsymbol{X}_i$ is obtained by

$$\boldsymbol{Q}_{X_i} = (\boldsymbol{A}_i^t\boldsymbol{W}_b\boldsymbol{A}_i)^{-1}\boldsymbol{A}_i^t\boldsymbol{W}_b\boldsymbol{Q}_{b_i}\boldsymbol{W}_b\boldsymbol{A}_i(\boldsymbol{A}_i^t\boldsymbol{W}_b\boldsymbol{A}_i)^{-1} \tag{3.9}$$

When estimating $\boldsymbol{X}_i$, other parameters are considered as non-stochastic constants. But their stochastic models are introduced after the first iteration since their results can be estimated by the least squares process. From Eq (3.4) we have

$$\boldsymbol{Q}_{b_i} = \boldsymbol{Q}_b + \boldsymbol{A}_J\boldsymbol{Q}_J\boldsymbol{A}_J^t \tag{3.10}$$

in which

$$\boldsymbol{A}_J = \begin{bmatrix}\boldsymbol{A}_1 & \boldsymbol{A}_2 & \Lambda & \boldsymbol{A}_k\end{bmatrix}_{without A_i} \tag{3.11}$$

and

$$\boldsymbol{Q}_J = \begin{bmatrix}\boldsymbol{Q}_{X_1} & & & \\ & \boldsymbol{Q}_{X_2} & & \\ & & \mathrm{O} & \\ & & & \boldsymbol{Q}_{X_k}\end{bmatrix}_{without Q_{X_i}} \tag{3.12}$$

Replacing $\boldsymbol{Q}_{bi}$ in Eq (3.9) from Eq (3.10) and expressing $(\boldsymbol{A}_i^t\boldsymbol{W}_b\boldsymbol{A}_i)^{-1}$ by $\boldsymbol{N}_i$ gives

$$\boldsymbol{Q}_{X_i} = \boldsymbol{N}_i + \boldsymbol{N}_i\boldsymbol{A}_i^t\boldsymbol{A}_J\boldsymbol{Q}_J\boldsymbol{A}_J^t\boldsymbol{A}_i\boldsymbol{N}_i \tag{3.13}$$

After each iteration the estimated results (the solution and the cofactor matrix) for all the unknown parameters will be updated. The results will finally converge to the same solutions obtained from the simultaneous LSE.

It is noticed that the full cofactor (covariance) matrix is not available with separate LSE. The cofactor matrix of the estimated parameters is given by

$$\boldsymbol{Q}_x = \begin{bmatrix} \boldsymbol{Q}_{X_1} & & & \\ & \boldsymbol{Q}_{X_2} & & \\ & & \text{O} & \\ & & & \boldsymbol{Q}_{X_k} \end{bmatrix} \qquad (3.14)$$

The correlations between the parameters which are not in the same group are not obtained.

If the separate LSE is compared with the iterative LSE (Hageman & Young 1981, Kok 1984, Phillips & Cornelius 1986) it is found that the solution given by the separate LSE is identical to that given by the Jacobi iteration or Gauss-Seidel iteration (Harley 1997). To show the equivalence of the two solutions, Eq (4.1) may be written as

$$(\boldsymbol{A}^t\boldsymbol{W}_b\boldsymbol{A})\boldsymbol{x} = \boldsymbol{A}^t\boldsymbol{W}_b\boldsymbol{b} \qquad (3.15)$$

i.e.,

$$\boldsymbol{N}\boldsymbol{x} = \boldsymbol{d} \qquad (3.16)$$

where

$$\boldsymbol{N} = \boldsymbol{A}^t\boldsymbol{W}_b\boldsymbol{A} \quad \text{and} \quad \boldsymbol{d} = \boldsymbol{A}^t\boldsymbol{W}_b\boldsymbol{b}\,.$$

In this case the solution of Eq (3.1) will be a least squares estimation. Dividing $\boldsymbol{x}$ into k groups as in Eq (3.2) and Eq (3.3), $\boldsymbol{N}$ and $\boldsymbol{d}$ become

$$\boldsymbol{N} = \begin{bmatrix} \boldsymbol{A}_1^t\boldsymbol{W}_b\boldsymbol{A}_1 & \boldsymbol{A}_1^t\boldsymbol{W}_b\boldsymbol{A}_2 & \Lambda & \boldsymbol{A}_1^t\boldsymbol{W}_b\boldsymbol{A}_k \\ \boldsymbol{A}_2^t\boldsymbol{W}_b\boldsymbol{A}_1 & \boldsymbol{A}_2^t\boldsymbol{W}_b\boldsymbol{A}_2 & \Lambda & \boldsymbol{A}_2^t\boldsymbol{W}_b\boldsymbol{A}_k \\ \Lambda & \Lambda & \Lambda & \Lambda \\ \boldsymbol{A}_k^t\boldsymbol{W}_b\boldsymbol{A}_1 & \boldsymbol{A}_k^t\boldsymbol{W}_b\boldsymbol{A}_2 & \Lambda & \boldsymbol{A}_k^t\boldsymbol{W}_b\boldsymbol{A}_k \end{bmatrix} \qquad (3.17)$$

and

$$\boldsymbol{d} = \begin{bmatrix} \boldsymbol{A}_1^t\boldsymbol{W}_b\boldsymbol{b} \\ \boldsymbol{A}_2^t\boldsymbol{W}_b\boldsymbol{b} \\ \text{M} \\ \boldsymbol{A}_k^t\boldsymbol{W}_b\boldsymbol{b} \end{bmatrix} \qquad (3.18)$$

If a Jacobi iteration is used, the solution of linear equations will be given by

$$\boldsymbol{X}_i^{[k+1]} = (\boldsymbol{A}_i^t\boldsymbol{W}_b\boldsymbol{A}_i)^{-1}(\boldsymbol{A}_i^t\boldsymbol{W}_b\boldsymbol{b} - \sum_{\substack{j=1 \\ j\neq i}}^{k} \boldsymbol{A}_i^t\boldsymbol{W}_b\boldsymbol{A}_j\boldsymbol{x}_j^{[k]}) \quad (3.19)$$

This solution is identical to that given by Eq (3.6).

Since matrix ($\boldsymbol{A}^t\boldsymbol{W}\boldsymbol{A}$) is symmetric and positive definite, this iterative process will always converge (Phillips & Cornelius 1986).

The separate LSE is an iterative process which estimates the unknown parameters iteratively and separately. But the iterations used here are different from the iterations used in the simultaneous LSE which are caused by the non-linear functional model. Even with a linear functional model iterations are still required by the separate LSE but not by the simultaneous method. The number of iterations required for the separate LSE is normally more than the simultaneous solution. But the total processing effort is small by comparison, especially in close range photogrammetry.

4. SEPARATE ADJUSTMENT OF CRPM

As discussed in the previous sections the separate least squares estimation is a technique of division, which divides the unknown parameters into groups. In photogrammetry, the unknown parameters are naturally divided into two groups, the coordinates of the object points and the camera parameters. The coordinates of the object points are the main requirement, while output of the camera parameters may not be necessary but have to be included as unknown parameters in the observation equations. The term *separate adjustment* can be used to describe the photogrammetric use of the separate LSE method (e.g. as the bundle adjustment is commonly used instead of the simultaneous LSE and the sequential adjustment instead of the sequential LSE).

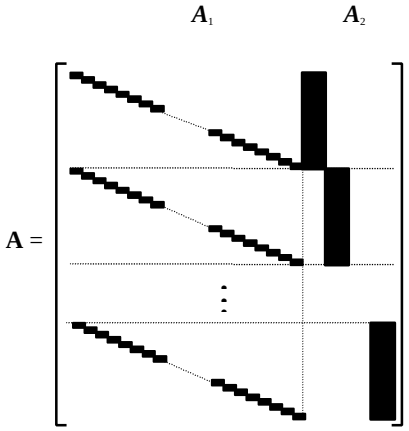
4.1 Free network separate adjustment

A free network adjustment means no constraints are involved in the adjustment process. The precision of the estimated results will be determined by the image observations only. Constraints are not necessary with the separate adjustment, but are normally required by the simultaneous bundle adjustment.

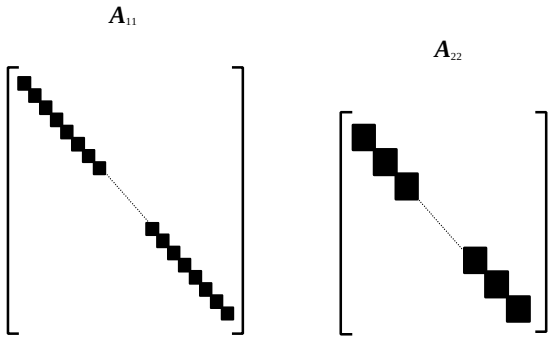
The separate adjustment is based on the same functional model used by the bundle adjustment, the collinearity equations. The linearized form is expressed as

$$\boldsymbol{A}_1\Delta\boldsymbol{x}_1 + \boldsymbol{A}_2\Delta\boldsymbol{x}_2 = \boldsymbol{b} \qquad (4.1)$$

where $\boldsymbol{x}_1$ denotes the coordinates of the object points and $\boldsymbol{x}_2$ the camera parameters. $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are design matrices and have the same formats as described in section 2. The structures of matrix $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}$ are very special as illustrated in Figure 4-1.



(a)



(b) (c)

Figure 4-1 (a) The structure of the design matrix $\boldsymbol{A}$ ($\boldsymbol{A}_1$ and $\boldsymbol{A}_2$)
(b) The structure of $\boldsymbol{A}_{11}$
(c) The structure of $\boldsymbol{A}_{22}$

The principle of the separate adjustment is to treat the unknown parameters $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ separately. The theory has been given in the previous section. In the separate adjustment process, only a part (group) of the parameters is adjusted in each step, either $\boldsymbol{x}_1$ or $\boldsymbol{x}_2$. The adjustment iterates between the two steps and the results will be the same as those given by the simultaneous bundle adjustment.

4.1.1 Adjusting the object points

When adjusting the object points, the camera parameters are considered as constants. So $\boldsymbol{D}\boldsymbol{x}_2 = 0$. Therefore the observation equations for estimating the coordinates of the object points become

$$\boldsymbol{A}_1\Delta\boldsymbol{x}_1 = \boldsymbol{b} \tag{4.2}$$

By least squares, the corrections of the 3D coordinates are estimated by

$$\begin{aligned}\Delta\boldsymbol{x}_1 &= (\boldsymbol{A}_1^t\boldsymbol{W}_l\boldsymbol{A}_1)^{-1}\boldsymbol{A}_1^t\boldsymbol{W}_l\boldsymbol{b}\\ &= \boldsymbol{A}_{11}^{-1}\boldsymbol{A}_1^t\boldsymbol{W}_l\boldsymbol{b}\end{aligned} \tag{4.3}$$

Since $\boldsymbol{A}_{11}$ is a block diagonal matrix, the inverse of $\boldsymbol{A}_{11}$ can be calculated by inverting a series of 3×3 matrices. The matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_{11}$ can be stored compactly as illustrated in Figure 4-1 and the products of the matrices are simplified. Each small block in $\boldsymbol{A}_1$ is a 2×3 submatrix and each small block in $\boldsymbol{A}_{11}$ is a 3×3 submatrix (n is the number of the object points and m is the number of the cameras).

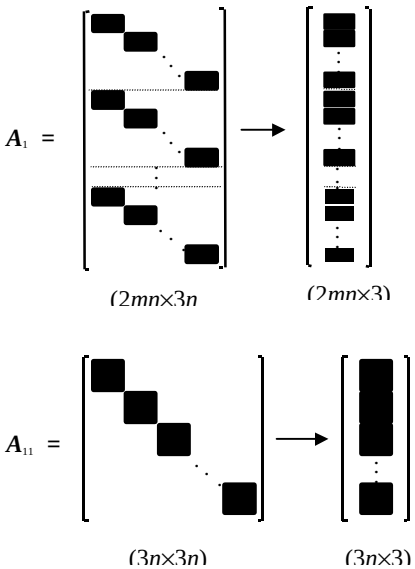


Figure 4-2 The structures of the matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_{11}$

Since the object points are independent of each other when the camera parameters are treated as constants, they can be adjusted separately. So the memory required can be reduced further. For the $\boldsymbol{i}$th object point, the size of $\boldsymbol{A}_{1i}$ is $2m$×3 and the size of $\boldsymbol{A}_{11i}$ is 3×3. The corrections of the 3D coordinates for the $\boldsymbol{i}$th object point are given by

$$\Delta\boldsymbol{x}_{1i} = \boldsymbol{A}_{11i}^{-1}\boldsymbol{A}_{1i}^t\boldsymbol{W}_{li}\boldsymbol{b}_i \tag{4.4}$$

Matrix $\boldsymbol{A}_{11i}$ can be calculated by a further partitioning according to the cameras, i.e.,

$$\begin{aligned}\boldsymbol{A}_{11i} &= \boldsymbol{A}_{1i}^t\boldsymbol{W}_{li}\boldsymbol{A}_{1i}\\ &= \sum_{j=1}^{m}\boldsymbol{A}_{1ij}^t\boldsymbol{W}_{lij}\boldsymbol{A}_{1ij}\\ &= \sum_{j=1}^{m}\boldsymbol{A}_{11ij}\end{aligned} \tag{4.5}$$

where $\boldsymbol{A}_{1ij}$ is a 2×3 matrix and $\boldsymbol{W}_{lij}$ is a 2×2 matrix, which are produced by the $\boldsymbol{i}$th point on the $\boldsymbol{j}$th camera. Similarly, $\boldsymbol{A}_{1i}^t\boldsymbol{W}_{li}\boldsymbol{b}_i$ can also be calculated by partitioning, i.e.,

$$\boldsymbol{A}_{1i}^t\boldsymbol{W}_{li}\boldsymbol{b}_i = \sum_{j=1}^{m}\boldsymbol{A}_{1ij}^t\boldsymbol{W}_{lij}\boldsymbol{b}_{ij} \tag{4.6}$$

where $\boldsymbol{b}_{ij}$ is a 2×1 matrix attributed to the $\boldsymbol{i}$th point on the $\boldsymbol{j}$th camera. In this case, the maximum size of the matrix required to obtain the corrections for the 3D coordinates of the object point is 3×3.

The computational time is directly proportional to the number of the object points. The minimum number of the cameras required is two.

4.1.2 Adjusting the cameras

When adjusting the camera parameters, the coordinates of the object points are considered as constants. So $\Delta\boldsymbol{x}_1 = 0$. Therefore the observation equations for estimating the camera parameters become

$$\boldsymbol{A}_2\Delta\boldsymbol{x}_2 = \boldsymbol{b} \tag{4.7}$$

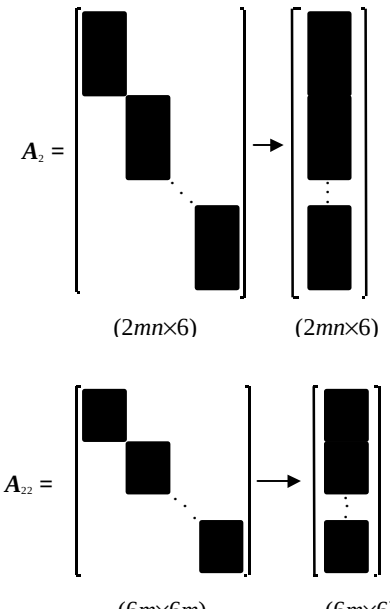


Figure 4-3 The structures of the matrices $\boldsymbol{A}_2$ and $\boldsymbol{A}_{22}$

By least squares, the corrections of the camera parameters are estimated by

$$\begin{aligned}\Delta \boldsymbol{x}_2 &= (\boldsymbol{A}_2^t \boldsymbol{W}_l \boldsymbol{A}_2)^{-1} \boldsymbol{A}_2^t \boldsymbol{W}_l \boldsymbol{b} \\ &= \boldsymbol{A}_{22}^{-1} \boldsymbol{A}_2^t \boldsymbol{W}_l \boldsymbol{b}\end{aligned} \tag{4.8}$$

where $\boldsymbol{A}_{22}$ is a block diagonal matrix, the inverse of $\boldsymbol{A}_{22}$ is calculated by inverting a series of 6×6 matrices. Because of their special structures, the matrices $\boldsymbol{A}_2$ and $\boldsymbol{A}_{22}$ can be stored compactly as illustrated in Figure 4-2 and the products of the matrices are simplified. Each small block in $\boldsymbol{A}_2$ is a $2n$×6 submatrix and each small block in $\boldsymbol{A}_{22}$ is a 6×6 submatrix.

Since the cameras are independent of each other, when the object points are fixed, the parameters of each individual camera can be adjusted separately. So the memory required can be reduced further. For the $\boldsymbol{j}$th camera, the size of $\boldsymbol{A}_{2j}$ is $2n$×6 and the size of $\boldsymbol{A}_{22j}$ is 6×6. The corrections of the parameters for the $\boldsymbol{j}$th camera are given by

$$\Delta \boldsymbol{x}_{2j} = \boldsymbol{A}_{22j}^{-1} \boldsymbol{A}_{2j}^t \boldsymbol{W}_{lj} \boldsymbol{b}_j \tag{4.9}$$

Matrix $\boldsymbol{A}_{22j}$ can be calculated by a further partitioning according to the object points, i.e.,

$$\begin{aligned}\boldsymbol{A}_{22j} &= \boldsymbol{A}_{2j}^t \boldsymbol{W}_{lj} \boldsymbol{A}_{2j} \\ &= \sum_{i=1}^{n} \boldsymbol{A}_{2ji}^t \boldsymbol{W}_{lji} \boldsymbol{A}_{2ji} \\ &= \sum_{i=1}^{n} \boldsymbol{A}_{22ji}\end{aligned} \tag{4.10}$$

where $\boldsymbol{A}_{2ji}$ is a 2×6 matrix and $\boldsymbol{W}_{lji}$ is a 2×2 matrix, which are produced by the $\boldsymbol{i}$th point on the $\boldsymbol{j}$th camera. Similarly, $\boldsymbol{A}_{2j}^t \boldsymbol{W}_{lj} \boldsymbol{b}_j$ can also be calculated by partitioning, i.e.,

$$\boldsymbol{A}_{2j}^t \boldsymbol{W}_{lj} \boldsymbol{b}_j = \sum_{i=1}^{n} \boldsymbol{A}_{2ji}^t \boldsymbol{W}_{lji} \boldsymbol{b}_{ji} \tag{4.11}$$

where $\boldsymbol{b}_{ji}$ is a 2×1 matrix attributed to the $\boldsymbol{i}$th point on the $\boldsymbol{j}$th camera. So the maximum size of the matrix required to obtain the corrections of the camera parameters is 6×6.

The computational time is directly proportional to the number of cameras. The minimum number of object points required is three.

4.1.3 Iteration between the two steps

The separate adjustment is an iterative process which is carried out between the two steps described above. After each iteration, the 3D coordinates and the camera parameters are refined. The iterative process terminates when the stop criterion (e.g. the maximum adjustment of the 3D coordinates is less than a given value) is met.

4.2 Datum definition

A datum must be defined in the simultaneous bundle adjustment to remove the column rank defects of the design matrix so the unknown parameters can be estimated. However the pre-definition of the datum is generally not necessary in the separate adjustment. No constraints are required to make the unknown parameters estimable when the separate adjustment is applied. The datum is held either by the camera exterior parameters when the coordinates of the object points are adjusted or by the object points when the camera parameters are adjusted. Since both the coordinates of the object points and the camera exterior parameters are related to the same coordinate system, the datum is actually determined by the starting values of the parameters. If the coordinates of the object points are adjusted first, the datum will be determined by the starting values of the camera parameters, otherwise it will be determined by the starting values of the object points.

Because of the uncertainty of the starting values, the results from the separate adjustment are in an arbitrary coordinate system if spatial controls are not applied. The results from the separate adjustment may not be numerically identical with that obtained from the traditional bundle adjustment due to the different datum definition. However the shape of the measured object is the same from both methods. So their results are equivalent. This can be verified by a rigid coordinate transformation of the object.

4.3 Precision estimation

The covariance (cofactor) matrix is a by-product which is normally produced directly by the simultaneous bundle adjustment at the same time when the unknown parameters are estimated. The square roots of the diagonal elements of the covariance matrix give the standard deviations to the corresponding parameters which are used to evaluate the precision of the measurement system. However, the full covariance matrix is not available directly from the separate adjustment. It can be calculated when it is required from the design matrix and the computation is very expensive in terms of time and memory. In many cases the full covariance matrix is not necessary for the purpose of the standard deviations of the estimated parameters.

From the separate adjustment, the cofactor matrix of the coordinates of the object points is given approximately by

$$\boldsymbol{Q}_{x_1} = (\boldsymbol{A}_1^t \boldsymbol{W}_l \boldsymbol{A}_1)^{-1} = \boldsymbol{A}_{11}^{-1} \tag{4.12}$$

For each object point a 3×3 cofactor matrix is given by

$$\boldsymbol{Q}_{x_{1i}} = (\boldsymbol{A}_{1i}^t \boldsymbol{W}_{li} \boldsymbol{A}_{1i})^{-1} = \boldsymbol{A}_{11i}^{-1} \tag{4.13}$$

and the covariance matrix the 3D coordinates is

$$\boldsymbol{C}_{x_{1i}} = \hat{\sigma}_0^2 \boldsymbol{Q}_{x_{1i}} \tag{4.14}$$

The cofactor matrix of the camera parameters is given approximately by

$$\boldsymbol{Q}_{x_2} = (\boldsymbol{A}_2^t \boldsymbol{W}_l \boldsymbol{A}_2)^{-1} = \boldsymbol{A}_{22}^{-1} \tag{4.15}$$

For each camera a 6×6 cofactor matrix is given by

$$\boldsymbol{Q}_{x_{2j}} = (\boldsymbol{A}_{2j}^{t}\boldsymbol{W}_{ij}\boldsymbol{A}_{2j})^{-1} = \boldsymbol{A}_{22j}^{-1} \qquad (4.16)$$

and the covariance matrix of the camera parameters is

$$\boldsymbol{C}_{x_{2j}} = \bar{s}_0^2\boldsymbol{Q}_{x_{2j}} \qquad (4.17)$$

For each object point the 3×3 covariance matrix $\boldsymbol{C}_{x_{1i}}$ is adequate to evaluate the precision of the estimated 3D coordinates and the error ellipsoid for each object point. For each camera the covariance matrix $\boldsymbol{C}_{x2j}$ is also available to analyse the precision and the correlations between the camera parameters.

The approximations are caused by the neglect of the variances of the camera parameters and the 3D coordinates of the object points when calculating $\boldsymbol{C}_{x_{1i}}$ and $\boldsymbol{C}_{x2j}$ respectively. These approximations can be compensated for by including the variances into the iterative process, but with more computational effort.

Simulation tests have shown that the approximations are quite acceptable for a multi-camera strong network especially in close range photogrammetry. The differences caused by the approximations were normally less than one percent.

4.4 Number of iterations

The number of iterations required for the separate adjustment process depends on the closeness of the starting values to their final results. However, more iterations are generally required for the separate adjustment than for the bundle adjustment. Normally four iterations are enough to give satisfactory results for the bundle adjustment with reasonable starting values, while for the separate adjustment ten or a few tens are required. Table 4-1 shows the maximum adjustment of the coordinates and the sum of squares of the residuals ($f = v^tWv$) after each iteration for a close range photogrammetric measurement network with 100 object points and 4 cameras.

Table 4-1 The adjusted results by the bundle adjustment and the separate adjustment

	Bundle adjustment		Separate adjustment	
No. of Iteration	Max. adjustment (mm)	$f = v^tWv$ (mm^2)	Max. adjustment (mm)	$f = v^tWv$ (mm^2)
1	19.7646	0.80576482	13.8760	1.24956487
2	0.3381	0.00375875	1.0679	0.12538475
3	0.0128	0.00011768	0.2812	0.02086453
4	0.0001	0.00011342	0.0778	0.00484658
5			0.0259	0.00103845
6			0.0125	0.00026584
7			0.0071	0.00011747
8			0.0042	0.00011375
9			0.0025	0.00011347
10			0.0014	0.00011344
11			0.0008	0.00011342
12			0.0005	0.00011342
13			0.0003	0.00011342
14			0.0001	0.00011342

4.5 Consistency with the bundle solution

The separate adjustment is based on the same functional model as the bundle adjustment. The target functions of the least squares from the two methods are also same, which are the sums of the weighted squares of residuals on the image planes. Simulation tests and practical tests show that the two methods always arrive at the same minimisation. Each individual residual for all the observations has also been checked and found to be the same for both methods. This means that their results are equivalent. The coordinates of their solution may not be numerically identical because of different datum definitions, but the shapes of the measured object from the two methods are always same. This has been verified by the rigid coordinate transformation and by including control points in the adjustment process.

4.6 Computational complexity

Least squares adjustment is an expensive computational process. Inverting the coefficient matrix $\boldsymbol{A}^t\boldsymbol{W}\boldsymbol{A}$ is the main cost in terms of speed and memory. In the simultaneous bundle adjustment, $\boldsymbol{A}^t\boldsymbol{W}\boldsymbol{A}$ is a symmetric positive definite matrix. Fast algorithms (for example, Cholesky) can be used to compute the inverse of $\boldsymbol{A}^t\boldsymbol{W}\boldsymbol{A}$. However even then the computational complexity is still high. If the size of $\boldsymbol{A}^t\boldsymbol{W}\boldsymbol{A}$ is $u\times u$, the time required for computing the inverse of $\boldsymbol{A}^t\boldsymbol{W}\boldsymbol{A}$ is directly proportional to u^3 and the memory needed is directly proportional to u^2.

Suppose m images are used to measure n object points in a close range photogrammetric measurement system. If the bundle adjustment with inner constraints on the object points is used, the size of the coefficient matrix $\boldsymbol{A}^t\boldsymbol{W}\boldsymbol{A}$ is $(3n+6m)\times(3n+6m)$. So the computational complexity for one iteration is (T = time, M = memory, B = bundle adjustment)

$$\mathsf{T}(\mathrm{B}) \propto (3n+6m)^3 \qquad (4.18)$$

and

$$\mathsf{M}(\mathrm{B}) \propto (3n+6m)^2 \qquad (4.19)$$

With the separate adjustment, the computational complexity of time for one iteration is (S = separate adjustment)

$$\mathsf{T}(\mathrm{S}) \propto m\,n \qquad (4.20)$$

and the maximum memory required is a 6×6 unit no matter how many object points and cameras are involved. The time required for the separate adjustment can be expressed as

$$t_s = C_s\,m\,n\,I \qquad (4.21)$$

where m is the number of the cameras, n is the number of the object points and I is the number of iterations. C_s is a coefficient which may vary according to the computers. It is found to be 215 μs for a SUN Sparc Classic and 42 μs for a 120 MHz Pentium.

5. CONCLUSIONS

Close range photogrammetry has been widely used in the areas where 3D coordinates are required. Least squares estimation methods have been successfully used to deal with redundant measurements from image observations. Conventionally, all the unknown parameters are estimated simultaneously in the least squares process. This leads to the bundle adjustment in close range photogrammetry, which is very expensive in terms of computation time and memory requirements. Methods such as sequential adjustment, unified bundle adjustment and Block Successive Over Relaxation can be use in some cases to improve the conventional bundle adjustment. But none of these methods is ideal for real-time measurement. Based on the theory of the separate least squares estimation, an alternative method, named *separate adjustment*, was developed and successfully used in close range photogrammetry to replace the conventional bundle adjustment. The number of iterations required for the separate adjustment may be more than that for the bundle adjustment. However, due to the simple computation and the linear computational complexity, the speed of convergence of the separate adjustment is much faster than that of the bundle adjustment, especially for large data sets from a convergent measurement network. The maximum memory required by the separate adjustment is limited to a 6×6 (or 14×14 when camera interior parameters are considered) unit no matter how many cameras and object points are involved. Because of the high speed and low memory requirements, the separate adjustment can be recommended for use in the real-time measurement.

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